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Slow flow through stationary random beds and suspensions of spheres

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Stokes flow through a random, moderately dense bed of spheres is treated by a generalization of Brinkman's (1947) method, which is applicable to both stationary beds and suspensions. For stationary beds, Darcy's law with a permeability result similar to Brinkman's is derived. For suspensions an effective viscosity $\mu/(1-2{\cdot}50\phi)$ is found, where ϕ is the volume fraction of spheres. Also, an expression for the settling velocity is derived.

1. Introduction

Brinkman (1947), considering the viscous force exerted on a dense swarm of particles by a fluid flowing through them, introduced a very nice idea. Since the force on a single particle in a slow stream is calculated from the Stokes-flow velocity field, and the flow through a swarm of particles is described by Darcy's empirical equation for flow through a porous mass, Brinkman reasoned that the force on a particle situated in a swarm of particles could be calculated as if it were a solid particle imbedded in a porous mass. He represented the porous mass by modifying Stokes's equation, adding a Darcy resistance term to it, so that the effect of all the other particles is treated in an average sense. This method has been received with some scepticism, because of the empirical nature of Darcy's equation. The method has been extended recently by Spielman & Goren (1968) to the flow through fibrous media.

Tam (1969) put Brinkman's method in better theoretical shape, by treating the swarm of particles as point forces in Stokes flow and ensemble averaging over all particle positions except that of the primary particle. The resulting equation is the same as Brinkman's and has the same basic fault. It is uncertain what one should use for the effective viscosity, the fluid viscosity, or a viscosity which accounts for the concentration of the particles as Einstein's correction does for dilute suspensions.

The present paper resolves the viscosity question and generalizes Brinkman's method, so that it is applicable to the flow of suspensions, where the determination of the effective viscosity is the central problem. A statistical formulation due to Saffman (1971) is extended, and adapted to the present problem. The basic resulting equation is a Stokes equation with a statistically defined resistance term. In addition to a complete formulation of the problem for dense systems, the main contribution in this paper is the observation, after approximations similar to Brinkman's, that the resistance term is not simply proportional to the velocity, but is of the form $A\mathbf{U}+B\nabla^2\mathbf{U}$, which combines with the Stokes terms to give an effective viscosity. One might have assumed the resistance of the more general form, $A\mathbf{U}+B\nabla^2\mathbf{U}+C\nabla^4\mathbf{U}+\ldots$, an infinite series in the operator ∇^2, but it turns out, and was anticipated, that $\nabla^4\mathbf{U}$ is proportional to $\nabla^2\mathbf{U}$, and so on. Therefore, the series can be summed to the form first cited. This is for a fixed bed. There is a similar but simpler conclusion for suspensions.

In § 2 the statistical development is shown. (Some readers may find a physically oriented statistical reference, like Beran 1968, helpful.) This leads to a somewhat different formulation of Brinkman's problem, and is compared with it in § 3, disregarding the viscosity uncertainty. In §§ 4 and 5 the complete problem for fixed beds and suspensions is solved.

2. Statistical formulation

Consider very slow flow through a porous medium consisting of rigidly fixed or freely suspended solid particles. (We shall restrict to spheres later.) Let the orientation and geometry of these solids be specified statistically from an ensemble of possible geometries. For each member of this ensemble, the velocity field $\mathbf{u}$ is determined by the solution of Stokes's equations,

$$\operatorname{div}\mathbf{u} = 0,$$
$$0 = -\nabla(p-\rho\mathbf{g}\,.\,\mathbf{r})+\mu\nabla^2\mathbf{u},$$

where p is the pressure and $\mathbf{g}$ is the gravitational force per unit mass. These equations are to be solved with $\mathbf{u} = \mathbf{u}_s$ on solid boundaries, where $\mathbf{u}_s$ is the surface velocity of the solids. $\mathbf{u}_s$ is zero for fixed particles and determined from a rigid-body motion for free particles, the translational velocity and angular velocity of the particles being found from conditions on the total force and torque.

It is convenient to define a function $H(\mathbf{r})$, following Saffman (1971), which is zero in solids and unity in fluid. H depends on the statistical parameters which specify the distribution of the solids. We denote by $\langle\,\rangle$ an ensemble average. The ensemble average of $H(\mathbf{r})$, with $\mathbf{r}$ fixed, is

$$\langle H\rangle = 1-\phi, \tag{2.1}$$

where ϕ is the volume concentration of solids ($1-\phi$ is the porosity). The average velocity at a point will be denoted by $\langle\mathbf{u}\rangle$. In taking this average, the point $\mathbf{r}$ will be in the fluid in some members of the ensemble and in solid in others. The velocity in the solids is either zero or given by the rigid-body motion of the particles. It is defined in either case. A mean velocity which is typical of the fluid, a mean interstitial velocity, may be defined by

$$\langle H\mathbf{u}\rangle/\langle H\rangle,$$

which is the same as averaging only on those members of the ensemble for which the point is in the fluid. The velocity $\langle\mathbf{u}\rangle$ is more useful, however. For stationary beds, it is the seepage velocity. Integrating it over an element of surface gives the average volume flow of the fluid across the surface. For suspensions it is the velocity of the composite material. Integrated over an element of surface, it gives the average volume flow of the composite.

The Newtonian stress,

$$T_{ij} = -(p-\rho\mathbf{g}.\mathbf{r})\,\delta_{ij}+\mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right), \tag{2.2}$$

is defined only in the fluid. An average stress,

$$\langle HT_{ij}\rangle = -\langle H(p-\rho\mathbf{g}.\mathbf{r})\rangle\,\delta_{ij}+\mu\left\langle H\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\right\rangle, \tag{2.3}$$

counts zero for points in the solid. Now use

$$\left\langle H\frac{\partial u_i}{\partial x_j}\right\rangle = \frac{\partial}{\partial x_j}\langle Hu_i\rangle-\left\langle u_i\frac{\partial H}{\partial x_j}\right\rangle \tag{2.4}$$

and

$$\left\langle (1-H)\frac{\partial u_i}{\partial x_j}\right\rangle = \frac{\partial}{\partial x_j}\langle(1-H)\,u_i\rangle+\left\langle u_i\frac{\partial H}{\partial x_j}\right\rangle. \tag{2.5}$$

In the last term in each of these relations, $\partial H/\partial x_j$ is a generalized function which is zero everywhere except at the solid boundary, where it is infinite. It is a unit normal times a Dirac delta function. On adding these two expressions, we find

$$\left\langle H\frac{\partial u_i}{\partial x_j}\right\rangle+\left\langle(1-H)\frac{\partial u_i}{\partial x_j}\right\rangle = \frac{\partial\langle u_i\rangle}{\partial x_j}. \tag{2.6}$$

Now add to this the same expression with i and j interchanged:

$$\left\langle H\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\right\rangle+\left\langle(1-H)\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\right\rangle = \frac{\partial\langle u_i\rangle}{\partial x_j}+\frac{\partial\langle u_j\rangle}{\partial x_i}. \tag{2.7}$$

The second term on the left averages only on those members of the ensemble which are in the solid. Since the velocity in the solids is at most rigid-body motion, $\partial u_i/\partial x_j+\partial u_j/\partial x_i$ is zero there, so this term vanishes identically. Therefore, (2.3) becomes

$$\langle HT_{ij}\rangle = -\langle H(p-\rho\mathbf{g}.\mathbf{r})\rangle\,\delta_{ij}+\mu\left(\frac{\partial\langle u_i\rangle}{\partial x_j}+\frac{\partial\langle u_j\rangle}{\partial x_i}\right). \tag{2.8}$$

The quantity $\langle H\mathbf{T}\rangle.\hat{\mathbf{n}}dS$ is the force which the *fluid* on one side of the element $\hat{\mathbf{n}}dS$ exerts on the *fluid* on the other side, since in the averaging, the stress is counted as zero in solids. Since the quantity

$$\overline{p} = \langle Hp\rangle/\langle H\rangle \tag{2.9}$$

is the mean static pressure in the fluid, (2.8) can be written

$$\langle HT_{ij}\rangle = -(1-\phi)\,(\overline{p}-\rho\mathbf{g}.\mathbf{r})\,\delta_{ij}+\mu\left(\frac{\partial\langle u_i\rangle}{\partial x_j}+\frac{\partial\langle u_j\rangle}{\partial x_i}\right). \tag{2.10}$$

We make use of these results in the following way. From the trace of (2.7) and the continuity equation, we find

$$\operatorname{div}\langle\mathbf{u}\rangle = 0. \tag{2.11}$$

From Stokes's equation, written as $\operatorname{div}\mathbf{T} = 0$, we find

$$0 = \langle H\operatorname{div}\mathbf{T}\rangle = \operatorname{div}\langle H\mathbf{T}\rangle-\langle\mathbf{T}.\nabla H\rangle. \tag{2.12}$$

Now use (2.8) and (2.11) to get

$$0 = -\nabla(1-\phi)\,(\overline{p}-\rho\mathbf{g}.\mathbf{r})+\mu\nabla^2\langle\mathbf{u}\rangle-\langle\mathbf{T}.\nabla H\rangle. \tag{2.13}$$

This compares with Saffman's (1971) equation (2.15). The last term on the right is the contribution from those members of the ensemble for which the point is on a solid boundary. It is a force density, the average force per unit volume which the solids exert on the fluid. This may be seen by integrating this term over a macroscopic volume. Making use of the delta function property of ∇H gives

$$-\int \langle \mathbf{T} . \nabla H \rangle \, d\mathbf{r} = -\left\langle \int \mathbf{T} . \hat{\mathbf{n}} \, dS \right\rangle, \tag{2.14}$$

where the surface integral is over all solid boundaries in the volume and $\hat{\mathbf{n}}$ is a unit normal directed into the fluid. The last integral is the average force which the solids in the volume exert on the fluid. In the form in which (2.13) is written each term represents a force per unit volume on the fluid. In particular, the first term is not $\nabla \overline{p}$ but $\nabla(1-\phi)\,\overline{p}$, $(1-\phi)$ being the fluid fraction per unit area on which $\overline{p}$ acts.

More progress can be made with a specific model for the porous medium. We will assume from now on that the medium is a collection of N identical spheres of radius a. Let the centres of these spheres be at $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N$. Let $P_N(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N)$ be a probability density for the location of the centres. For each member of the ensemble, the velocity field and H are functions of all of the $\mathbf{r}_j$'s. Ensemble averages are then computed from

$$\langle G \rangle = \int G(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N)\, P_N(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N)\, d\mathbf{r}_1 d\mathbf{r}_2 \ldots d\mathbf{r}_N, \tag{2.15}$$

where G is a representative function. A probability density for a typical single particle is defined by

$$P_1(\mathbf{r}_1) = \int P_N(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N)\, d\mathbf{r}_2 d\mathbf{r}_3 \ldots d\mathbf{r}_N, \tag{2.16}$$

i.e. by integrating over the positions of all the other particles. For an 'overlapping sphere' model

$$P_N = P_1(\mathbf{r}_1)\, P_1(\mathbf{r}_2) \ldots P_1(\mathbf{r}_N).$$

A more realistic probability density must account for the mutual exclusiveness of the spheres. Since the spheres are identical, the number density n (defined such that $n d\mathbf{r}$ is the ensemble average of the number of particles with centres in volume $d\mathbf{r}$) is given by

$$n(\mathbf{r}) = N P_1(\mathbf{r}). \tag{2.17}$$

The function H is given explicitly by

$$H(\mathbf{r}; \mathbf{r}_1, \mathbf{r}_2, \ldots \mathbf{r}_N) = 1 - \sum_{j=1}^{N} \mathscr{H}(a - |\mathbf{r} - \mathbf{r}_j|), \tag{2.18}$$

where $\mathscr{H}$ is the Heaviside step function, defined to be zero when its argument is negative, one otherwise. With this expression for H we get

$$\begin{aligned} \langle H \rangle &= 1 - \phi \\ &= \int H P_N \, d\mathbf{r}_1 d\mathbf{r}_2 \ldots d\mathbf{r}_N \\ &= 1 - N \int \mathscr{H}(a - |\mathbf{r} - \mathbf{r}_1|)\, P_1(\mathbf{r}_1)\, d\mathbf{r}_1 \\ &= 1 - N \int_{|\mathbf{r}-\mathbf{r}_1|<a} P_1(\mathbf{r}_1)\, d\mathbf{r}_1 \\ &= 1 - \int_{|\mathbf{r}-\mathbf{r}_1|<a} n(\mathbf{r}_1)\, d\mathbf{r}_1, \end{aligned} \tag{2.19}$$

relating the porosity to the number density. If the particles are uniformly distributed, so that n is uniform, this gives

$$\phi = \tfrac{4}{3}\pi a^3 n, \tag{2.20}$$

an obvious result.

We shall also need to consider conditional probability densities. Suppose from the ensemble of particles we single out the sub-ensemble for which one particle, particle 1 say, has the same position in each member. In this sub-ensemble, particle 1 is a common solid boundary. The probability density for the remaining particles, a conditional probability, is

$$P_N(\mathbf{r}_1, \mathbf{r}_2, \ldots \mathbf{r}_N)/P_1(\mathbf{r}_1). \tag{2.21}$$

We shall denote sub-ensemble averages by an angle bracket with subscript 1:

$$\langle G \rangle_1 = \int G(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N) \frac{P_N(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N)}{P_1(\mathbf{r}_1)} d\mathbf{r}_2 d\mathbf{r}_3 \ldots d\mathbf{r}_N. \tag{2.22}$$

The following items are of interest. The probability density of a typical particle in the sub-ensemble, particle 2 say, is

$$P_2(\mathbf{r}_1, \mathbf{r}_2)/P_1(\mathbf{r}_1), \tag{2.23}$$

and the conditional number density is

$$n_1(\mathbf{r}_2) = NP_2(\mathbf{r}_1, \mathbf{r}_2)/P_1(\mathbf{r}_1). \tag{2.24}$$

For particles which are randomly distributed except for the constraint of mutual exclusiveness, $P_2(\mathbf{r}_1, \mathbf{r}_2)$ is a function which is zero when $|\mathbf{r}_1 - \mathbf{r}_2| < 2a$, and equals $P_1(\mathbf{r}_1)\, P_1(\mathbf{r}_2)$ when the particles are not this close. The density n_1 is then simply an empty sphere of radius $2a$ surrounding particle 1, and uniform density n outside this. More complicated distributions can arise with interacting particles, where present position and densities of particles are determined by their motion, and the distribution function cannot be chosen at will but must satisfy an appropriate modified Liouville equation.

Referring now to (2.13), the term $\langle \mathbf{T} . \nabla H \rangle$ can be put in more explicit form by using

$$\begin{aligned} \nabla H &= \nabla\left(1 - \sum_{j=1}^{N} \mathscr{H}(a - |\mathbf{r} - \mathbf{r}_j|)\right) \\ &= \sum_{j=1}^{N} \frac{\mathbf{r} - \mathbf{r}_j}{|\mathbf{r} - \mathbf{r}_j|} \delta(|\mathbf{r} - \mathbf{r}_j| - a), \end{aligned} \tag{2.25}$$

where δ is the Dirac delta function. Then

$$\begin{aligned} \langle \mathbf{T} . \nabla H \rangle &= \sum_{j=1}^{N} \int P_N(\mathbf{r}_1, \ldots, \mathbf{r}_N)\, \delta(|\mathbf{r} - \mathbf{r}_j| - a)\, \mathbf{T} . \frac{\mathbf{r} - \mathbf{r}_j}{|\mathbf{r} - \mathbf{r}_j|} d\mathbf{r}_1 \ldots d\mathbf{r}_N \\ &= N \int P_N \delta(|\mathbf{r} - \mathbf{r}_1| - a) \mathbf{T} . \frac{\mathbf{r} - \mathbf{r}_1}{|\mathbf{r} - \mathbf{r}_1|} d\mathbf{r}_1 \ldots d\mathbf{r}_N \\ &= \int n(\mathbf{r}_1)\, \delta(|\mathbf{r} - \mathbf{r}_1| - a)\, \langle \mathbf{T} \rangle_1 . \frac{\mathbf{r} - \mathbf{r}_1}{|\mathbf{r} - \mathbf{r}_1|} d\mathbf{r}_1, \end{aligned} \tag{2.26}$$

where the second line follows because P_N is invariant to interchange of particles, the particles being identical. In the third line, we have used the definition of number density and conditional average. Finally, we note that because of the delta function, the volume integral may be converted to a surface integral. Denoting by $\hat{\mathbf{n}} = \mathbf{r}-\mathbf{r}_1/|\mathbf{r}-\mathbf{r}_1|$, the unit outward normal to the particle, the only points which contribute to the integral are $\mathbf{r}_1 = \mathbf{r}-a\hat{\mathbf{n}}$ for all $\hat{\mathbf{n}}$ on the unit sphere. Let $d\hat{\mathbf{n}}$ be an element of solid angle on the unit sphere. Then

$$\langle \mathbf{T}.\nabla H\rangle = \int_{\mathbf{r}_1=\mathbf{r}-a\hat{\mathbf{n}},\,\mathbf{r}\text{ fixed}} n(\mathbf{r}-a\hat{\mathbf{n}})\langle \mathbf{T}\rangle_1.\hat{\mathbf{n}}a^2\,d\hat{\mathbf{n}}. \tag{2.27}$$

This requires some explanation. Recall that $\mathbf{r}$ is fixed in ensemble averaging. $\langle \mathbf{T}\rangle_1.\hat{\mathbf{n}}$ is the average traction at $\mathbf{r}$ when a particle is known to be centred at $\mathbf{r}_1$. (It is a function of two arguments $\mathbf{r}_1$ and $\mathbf{r}$.) Since $\mathbf{r}_1 = \mathbf{r}-a\hat{\mathbf{n}}$, the point $\mathbf{r}$ is on the surface of this sphere. The integration, then, is a weighted (by n) average traction at a point, the average being over the positions of all spheres which touch the point. In the case of most interest, the number density is uniform, and (2.13) becomes

$$0 = \nabla(1-\phi)(\overline{p}-\rho\mathbf{g}.\mathbf{r})+\mu\nabla^2\langle\mathbf{u}\rangle - na^2\int_{\mathbf{r}_1=\mathbf{r}-a\hat{\mathbf{n}},\,\mathbf{r}\text{ fixed}}\langle \mathbf{T}\rangle_1.\hat{\mathbf{n}}\,d\hat{\mathbf{n}}. \tag{2.28}$$

(In integrals like this, we shall always indicate both arguments of $\langle \mathbf{T}\rangle_1$, by subscripts on the integral.) To compute $\langle \mathbf{T}\rangle_1$, we have, by a modification of (2.10),

$$\langle T_{ij}\rangle = -(\overline{p}_1-\rho\mathbf{g}.\,\mathbf{r})+\mu\left(\frac{\partial\langle u_i\rangle_1}{\partial x_j}+\frac{\partial\langle u_j\rangle_1}{\partial x_i}\right), \tag{2.29}$$

(since $\phi_1 = 0$ at the surface of the sphere), which requires the average static pressure and velocity when particle 1 has a known position.

An equation for $\langle\mathbf{u}\rangle_1$ may be obtained by the above process, using a conditional probability in place of P_N. The equivalent of (2.13) is

$$0 = -\nabla(1-\phi_1)(\overline{p}_1-\rho\mathbf{g}.\mathbf{r})+\mu\nabla^2\langle\mathbf{u}\rangle_1-\langle \mathbf{T}.\nabla H\rangle_1. \tag{2.30}$$

The equivalent of (2.27) is

$$\langle \mathbf{T}.\nabla H\rangle_1 = \int_{\mathbf{r}_2=\mathbf{r}-a\hat{\mathbf{n}},\,\mathbf{r}_1\text{ fixed},\,\mathbf{r}\text{ fixed}} n_1(\mathbf{r}-a\hat{\mathbf{n}})\langle \mathbf{T}\rangle_{1,2}.\hat{\mathbf{n}}a^2 d\hat{\mathbf{n}}, \tag{2.31}$$

where particle 1 is fixed, and the centre of particle 2, the field particle, is at $\mathbf{r}-a\hat{\mathbf{n}}$. This differs from (2.27), because of the extremely non-uniform density field n_1 caused by the presence of particle 1. $n_1(\mathbf{r}-a\hat{\mathbf{n}})$ is zero for the range of $\hat{\mathbf{n}}$ for which $|\mathbf{r}-\mathbf{r}_1-a\hat{\mathbf{n}}| < 2a$, i.e. for all those positions of particle 2 that are excluded by the presence of particle 1. The stress $\langle \mathbf{T}\rangle_{1,2}$ that occurs here depends on $\langle\mathbf{u}\rangle_{1,2}$, the average velocity field when the positions of two particles are known. We can continue this process, deriving an equation for $\langle\mathbf{u}\rangle_{1,2}$. It would have a resistance term which depends on the average velocity when three particles are fixed, $\langle\mathbf{u}\rangle_{1,2,3}$, and so on. This leads to a hierarchy of modified Stokes equations.

We shall close this hierarchy of equations by a simple proposal, which generalizes an idea due to Brinkman (1947). Restricting to uniform number density, we assume that

$$na^2\int_{\mathbf{r}_1=\mathbf{r}-a\hat{\mathbf{n}},\,\mathbf{r}\text{ fixed}}\langle \mathbf{T}\rangle_1.\hat{\mathbf{n}}\,d\hat{\mathbf{n}} = \mathscr{F}(\langle\mathbf{u}\rangle), \tag{2.32}$$

where $\mathscr{F}(\langle \mathbf{u}\rangle)$ is a linear functional of $\langle \mathbf{u}\rangle$, to be determined. Then $\langle \mathbf{u}\rangle$ must satisfy

$$\left.\begin{aligned} \operatorname{div}\langle \mathbf{u}\rangle &= 0, \\ 0 &= -\nabla(\bar{p}-\rho\mathbf{g}.\mathbf{r})+\frac{\mu}{1-\phi}\nabla^2\langle \mathbf{u}\rangle-\frac{1}{1-\phi}\mathscr{F}(\langle \mathbf{u}\rangle), \end{aligned}\right\} \tag{2.33}$$

where ϕ is constant because of the uniform number density. Further, we assume that $\langle \mathbf{u}\rangle_1$ and $\bar{p}_1$ satisfy the very same equations, with the same constant ϕ, as if particle 1 were a solid inclusion in a porous material. Since $\langle \mathbf{u}\rangle_1$ *should* satisfy (2.30), it is clear that several approximations have been made. First of all ϕ_1 has been replaced by ϕ neglecting the fact that particle 1 excludes all spheres which would overlap it. ϕ_1 should be zero in a sphere of radius $2a$ surrounding particle 1.

Second, we have assumed that $\langle \mathbf{T}.\nabla H\rangle_1$, given by (2.31) is the same functional of $\langle \mathbf{u}\rangle_1$ as $\langle \mathbf{T}.\nabla H\rangle$ is of $\langle \mathbf{u}\rangle$. This requires that n_1 be replaced by n, again neglecting the mutual exclusiveness of the particles. While these are rather heavy approximations, they are at least pinpointed here, and the resulting imagery is clear.

The problem now is to solve (2.33) for $\langle \mathbf{u}\rangle_1$ and $\bar{p}_1$ with boundary conditions on the surface of a sphere centred at $\mathbf{r}_1$, and such that $\langle \mathbf{u}\rangle_1$ tends to the unperturbed velocity $\langle \mathbf{u}\rangle$ far from the sphere. We assume that $\mathbf{r}_1$ is not near external boundaries. With the solution to this problem, we compute the stress at the surface of the sphere, and use this to find $\mathscr{F}(\langle \mathbf{u}\rangle)$ from (2.32). The integral will be a functional of the unperturbed velocity $\langle \mathbf{u}\rangle$ because of the boundary conditions on $\langle \mathbf{u}\rangle_1$. This is not so simple, because the solution itself depends on $\mathscr{F}$, which is not yet known. Determining $\mathscr{F}$ is part of the problem.

There is another difficulty that comes up here. In light of the approximations made, how should $\langle \mathbf{T}\rangle_1$ at the surface of the sphere be computed? Before we made any approximations, particle 1 was surrounded by a void region, and $\langle \mathbf{T}\rangle_1$ was given by (2.29). In the new imagery, this void region is shrunk to zero, and we approach the solid boundary from the outside through the porous material. The question of what stress to use is related to the jump in stress across a discontinuity in porosity. We will find that for fixed spheres

$$\mathscr{F}(\langle \mathbf{u}\rangle) = A\langle \mathbf{u}\rangle+B\nabla^2\langle \mathbf{u}\rangle,$$

and for suspensions of spheres $\mathscr{F}(\langle \mathbf{u}\rangle) = B\nabla^2\langle \mathbf{u}\rangle+C\mathbf{g}$. In either case, (2.33) will have a term with an effective viscosity $\tilde{\mu}$, which differs from the fluid viscosity. We shall assume that the stress at a solid inclusion is given by

$$\langle T_{ij}\rangle = -\bar{p}\delta_{ij}+\tilde{\mu}\left(\frac{\partial\langle u_i\rangle}{\partial x_j}+\frac{\partial\langle u_j\rangle}{\partial x_i}\right), \tag{2.34}$$

using this effective viscosity. Brinkman made the same assumption. It will be demonstrated in § 5 that this is consistent for a suspension of spheres, and it will be discussed further in appendix A in relation to systems with non-uniform porosity.

3. Comparison with the Brinkman formulation

Brinkman (1947) considered the slow flow through a swarm of fixed stationary spheres of uniform number density n. In order to compute the resistance in the empirical Darcy equation

$$0 = -\nabla\bar{p} - \frac{\mu}{k}\langle \mathbf{u}\rangle + \rho\mathbf{g}, \tag{3.1}$$

where k is the permeability, Brinkman proposed that the 'resistance' per unit volume, $\mu\langle\mathbf{u}\rangle/k$, be computed from

$$\frac{\mu}{k}\langle \mathbf{u}\rangle = n\mathbf{D}, \tag{3.2}$$

where $\mathbf{D}$ is the drag per particle.† The drag is to be computed from a Stokes–Darcy flow past a typical sphere, thus accounting for the presence of the other spheres. The problem is to solve

$$\left.\begin{aligned} \operatorname{div}\langle \mathbf{u}\rangle_1 &= 0, \\ 0 &= -\nabla\bar{p}_1 + \tilde{\mu}\nabla^2\langle\mathbf{u}\rangle_1 - \tilde{\mu}\alpha^2\langle\mathbf{u}\rangle_1, \end{aligned}\right\} \tag{3.3}$$

where $\tilde{\mu}$ is an effective viscosity, arbitrarily chosen and $\tilde{\mu}\alpha^2 = \mu/k$. These are to be solved with boundary conditions $\langle\mathbf{u}\rangle_1 \to \mathbf{U}$ (a constant) as $|\mathbf{r}-\mathbf{r}_1| \to \infty$. From the solution the drag on the spheres is calculated from

$$\mathbf{D} = a^2 \int_{\mathbf{r}_1 \text{ fixed}, \, \mathbf{r}=\mathbf{r}_1+a\hat{\mathbf{r}}} \langle \mathbf{T}\rangle_1 . \hat{\mathbf{r}}\, d\hat{\mathbf{r}}, \tag{3.4}$$

with the stress calculated from (2.34) with the effective viscosity $\tilde{\mu}$. Thus

$$\mathbf{D} = 6\pi\tilde{\mu}a(1+\alpha a + \tfrac{1}{3}\alpha^2 a^2)\,\mathbf{U}. \tag{3.5}$$

Using $n\mathbf{D} = \tilde{\mu}\alpha^2\mathbf{U}$ gives a quadratic equation for α:

$$\tfrac{9}{2}\phi(1+\alpha a + \tfrac{1}{3}\alpha^2 a^2) = \alpha^2 a^2, \tag{3.6}$$

where $\phi = 4\pi a^3 n/3$ is the volume fraction of spheres. The positive root of this is then used to calculate the permeability from $k = \mu/\tilde{\mu}\alpha^2$. It is noted from the quadratic that, for small ϕ, $\alpha^2 = 9\phi/2a^2$, so for a dilute situation $k = k_0 = 2a^2/9\phi$, assuming that $\tilde{\mu} \to \mu$ as $\phi \to 0$. (This is the same as calculating the resistance from Stokes drag.) The final result is presented as

$$\frac{k}{k_0} = \frac{\mu}{\tilde{\mu}}\{1 + \tfrac{3}{4}[\phi - (8\phi - 3\phi^2)]^{\frac{1}{2}}\}. \tag{3.7}$$

This depends explicitly on the effective viscosity, which is completely unknown, except it must tend to μ when ϕ is small. Brinkman takes $\tilde{\mu} = \mu$.

The resulting expression agrees with experiments made on flow through randomly packed beds, but these are necessarily performed at very high volume concentrations. Experiments for small and moderate ϕ are not completely satisfactory and are of two kinds. Measurements of flow through cubic arrays of

† Referring to (2.13), it appears that, if (3.1) is written

$$0 = (1-\phi)(-\nabla\bar{p} + \rho\mathbf{g}) + (1-\phi)\mu\langle\mathbf{u}\rangle/k,$$

then the last term can be properly called the resistance per unit volume. Brinkman should have used $(1-\phi)\mu\langle\mathbf{u}\rangle/k = n\mathbf{D}$ instead of (3.2). This gives a different result.

spherical beads on wires have been performed by Happel & Epstein (1954). While these are not randomly packed, they do agree quite well with the Brinkman formula, as seen in figure 1. There are also fluidization and sedimentation experiments which have random beds of spheres with the right concentration range, but these are not fixed beds. Experiments of this kind are not in good agreement with the Brinkman result. This will be discussed later.

We want to compare Brinkman's result with the corresponding result using the formulation of the last section but without calculating the effective viscosity. We will assume that the resistance term is directly proportional to the velocity. The problem is to solve

$$0 = -\nabla\bar{p}_1 + \frac{\mu}{1-\phi}\nabla^2\langle\mathbf{u}\rangle_1 - \frac{na^2}{1-\phi}\int_{\mathbf{r}_1=\mathbf{r}-a\hat{\mathbf{n}},\ \mathbf{r}\text{ fixed}}\langle\mathbf{T}\rangle_1.\hat{\mathbf{n}}\,d\hat{\mathbf{n}}. \tag{3.8}$$

Let $\tilde{\mu} = \mu/(1-\phi)$ and assume

$$\frac{na^2}{1-\phi}\int_{\mathbf{r}_1=\mathbf{r}-a\hat{\mathbf{n}},\ \mathbf{r}\text{ fixed}}\langle\mathbf{T}\rangle_1.\hat{\mathbf{n}}\,d\hat{\mathbf{n}} = \tilde{\mu}\alpha^2\langle\mathbf{u}\rangle_1, \tag{3.9}$$

then the problem is the same as Brinkman's except for the difference in computing the resistance and the different effective viscosity. At first sight one might think that, for uniform flow past a sphere, the drag integral and the average traction at a point should be the same, that the surface stress should be independent of the spatial position of the sphere. This is not true, because the pressure in the unperturbed uniform flow is not uniform, but is given by

$$\bar{p}_0 = -\tilde{\mu}\alpha^2\mathbf{U}.\mathbf{r}. \tag{3.10}$$

A term like this contributes to the drag,

$$a^2\int_{\mathbf{r}_1\text{ fixed},\ \mathbf{r}=\mathbf{r}_1+a\hat{\mathbf{r}}}\bar{p}_0(\mathbf{r})\,\hat{\mathbf{r}}\,d\hat{\mathbf{r}} = a^2\int\bar{p}_0(\mathbf{r}_1+a\hat{\mathbf{r}})\,\hat{\mathbf{r}}\,d\hat{\mathbf{r}} = -\tfrac{4}{3}\pi a^3\tilde{\mu}\alpha^2\mathbf{U}, \tag{3.11}$$

while the contribution to the mean traction at a point is zero:

$$a^2\int_{\mathbf{r}_1=\mathbf{r}-a\hat{\mathbf{n}},\ \mathbf{r}\text{ fixed}}\bar{p}_0(\mathbf{r})\,\hat{\mathbf{n}}\,d\hat{\mathbf{n}} = 0, \tag{3.12}$$

since the unperturbed pressure is independent of the location of the sphere. The velocity and the perturbation pressure, and hence the perturbation stress, are independent of the spatial position and contribute the same in each case. The mean traction at a point is therefore found to be

$$a^2\int_{\mathbf{r}_1=\mathbf{r}-a\hat{\mathbf{n}},\ \mathbf{r}\text{ fixed}}\langle\mathbf{T}\rangle_1.\hat{\mathbf{n}}\,d\hat{\mathbf{n}} = 6\pi\tilde{\mu}a[1+\alpha a+\tfrac{1}{3}\alpha^2a^2]\,\mathbf{U} - \tfrac{4}{3}\pi\tilde{\mu}a\alpha^2a^2\mathbf{U}, \tag{3.13}$$

the first term being Brinkman's drag, the second the drag due to the unperturbed pressure†. When set equal to $(1-\phi)\,\tilde{\mu}\alpha^2\mathbf{U}/n$, the following quadratic equation results:

$$\tfrac{9}{2}\phi([1+\alpha a+\tfrac{1}{3}\alpha^2a^2]-\tfrac{2}{9}\alpha^2a^2) = (1-\phi)\,\alpha^2a^2,$$

which simplifies to give the same results as (3.6). Therefore the permeability is given by (3.7), as before. We see that, if we use the same effective viscosity, we

† We shall continue to call the integral in (3.13) the 'mean traction at a point', though it is really the mean traction times the area of a sphere.

get the same result, despite the differing resistance calculations. However, the effective viscosity indicated in (3.8) is $\tilde{\mu} = \mu/(1-\phi)$, so this result differs from Brinkman's. These results are compared in figure 1 (along with the modification suggested in the above footnote, which gives (3.7) with ϕ replaced by $\phi/(1-\phi)$. It should be stressed that the effective viscosities used here are quite arbitrary. An elaborate calculation, performed in §4, will produce a rationally determined effective viscosity.

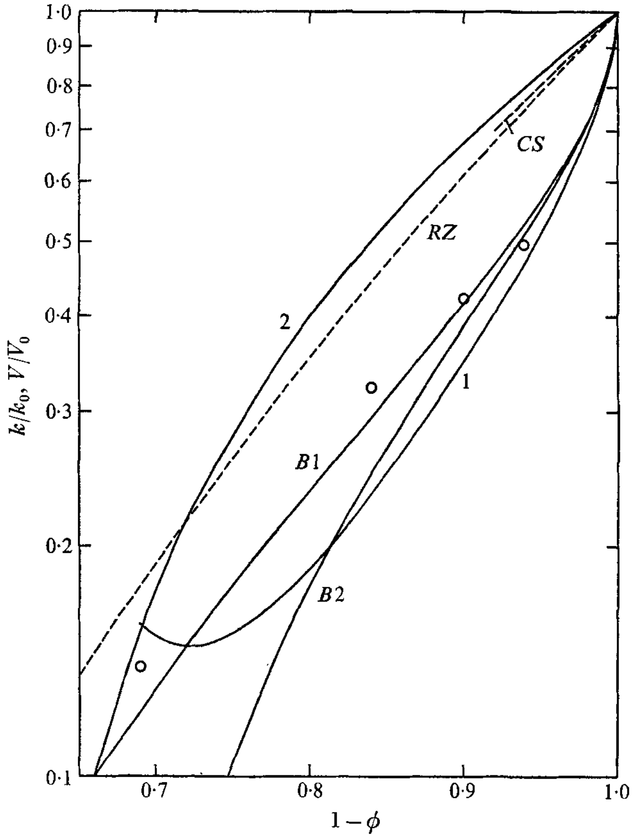

FIGURE 1. Permeability k/k_0 or settling velocity V/V_0 in random bed of spheres *vs.* void fraction $1-\phi$. $B1$, Brinkman (1947) formula; $B2$, modification of Brinkman formula according to the first footnote to §3; 1, permeability calculated from (4.46); 2, settling velocity calculated from (5.40); ○, experiments of Happel & Epstein (1954). *RZ*, sedimentation experiments of Richardson & Zaki (1954); *CS*, ultracentrifuge experiments of Cheng & Schachman (1955).

4. The flow through a fixed bed of spheres: calculation of the effective viscosity

We shall consider the same problem as in §3, the flow through a random bed of fixed spheres of uniform number density. The procedure will be almost the same, except that we shall let the unperturbed flow be arbitrary instead of uniform. If we assume that the resistance term is proportional to the velocity, as in

§3, and then compute either the drag or the mean traction at a point, we get a linear combination of $\langle \mathbf{u} \rangle$ and $\nabla^2 \langle \mathbf{u} \rangle$, evaluated at the centre of the sphere in the former case, or at the fixed point in the latter case. That is, it does not come out as assumed. This suggests, and it is borne out by calculation, that the resistance is the sum of two such terms. Calculating the mean traction determines both coefficients. We shall find both a Darcy term and an effective viscosity.

We anticipate

$$na^2 \int_{\mathbf{r}_1 = \mathbf{r} - a\hat{\mathbf{n}},\ \mathbf{r}\ \text{fixed}} \langle \mathbf{T} \rangle_1 . \hat{\mathbf{n}}\, d\hat{\mathbf{n}} = A\langle \mathbf{u} \rangle + B\nabla^2 \langle \mathbf{u} \rangle = \mathscr{F}(\langle \mathbf{u} \rangle). \tag{4.1}$$

We shall show that this is consistent, and determine A and B. With this assumption (2.33) becomes

$$0 = -\nabla(\bar{p} - \rho \mathbf{g} . \mathbf{r}) + \frac{\mu - B}{1-\phi} \nabla^2 \langle \mathbf{u} \rangle - \frac{A}{1-\phi} \langle \mathbf{u} \rangle. \tag{4.2}$$

We introduce the notation,

$$\frac{\mu - B}{1-\phi} = \tilde{\mu}, \quad \frac{A}{1-\phi} = \tilde{\mu}\alpha^2, \tag{4.3), (4.4}$$

and call the unperturbed velocity field $\langle \mathbf{u} \rangle = \mathbf{U}$ and the unperturbed pressure plus gravitational potential $\bar{p} - \rho \mathbf{g} . \mathbf{r} = \bar{p}_0$. The unperturbed flow satisfies

$$\left.\begin{aligned} \operatorname{div} \mathbf{U} &= 0, \\ 0 &= -\nabla \bar{p}_0 + \tilde{\mu}\nabla^2 \mathbf{U} - \tilde{\mu}\alpha^2 \mathbf{U}. \end{aligned}\right\} \tag{4.5}$$

With one sphere fixed we have, by assumption,

$$\left.\begin{aligned} \operatorname{div} \langle \mathbf{u} \rangle_1 &= 0, \\ 0 &= -\nabla(\bar{p}_1 - \rho \mathbf{g} . \mathbf{r}) + \tilde{\mu}\nabla^2 \langle \mathbf{u} \rangle_1 - \tilde{\mu}\alpha^2 \langle \mathbf{u} \rangle_1, \end{aligned}\right\} \tag{4.6}$$

to be solved with boundary conditions, $\langle \mathbf{u} \rangle_1 = 0$ on $\mathbf{r} = \mathbf{r}_1 + a\hat{\mathbf{r}}$ and $\langle \mathbf{u} \rangle_1 \to \mathbf{U}$ as $\mathbf{r} \to \infty$. Now let

$$\left.\begin{aligned} \langle \mathbf{u} \rangle_1 &= \mathbf{U} + \mathbf{v}, \\ \bar{p}_1 - \rho \mathbf{g} . \mathbf{r} &= \bar{p}_0 + p. \end{aligned}\right\} \tag{4.7}$$

Then, since $\mathbf{U}$ and $\bar{p}_0$ satisfy the same equations, we have

$$\left.\begin{aligned} \operatorname{div} \mathbf{v} &= 0, \\ 0 &= -\nabla p + \tilde{\mu}\nabla^2 \mathbf{v} - \tilde{\mu}\alpha^2 \mathbf{v}, \end{aligned}\right\} \tag{4.8}$$

with boundary conditions $\mathbf{v} = -\mathbf{U}(\mathbf{r}_1 + a\hat{\mathbf{r}})$ on the sphere, and $\mathbf{v} \to 0$ at infinity. With the solution of this we must compute

$$a^2 \int_{\mathbf{r}_1 = \mathbf{r}_0 - a\hat{\mathbf{n}},\ \mathbf{r}_0\ \text{fixed}} \langle \mathbf{T} \rangle_1 . \hat{\mathbf{n}}\, d\hat{\mathbf{n}}. \tag{4.9}$$

(We shall let the fixed point be $\mathbf{r}_0$ to avoid confusion with $\mathbf{r}$, the running variable.) The stress due to the unperturbed flow contributes nothing to this, since the stress tensor at a point is independent of the orientation of the surface element. Therefore we can use

$$\langle T_{ij} \rangle = -p\delta_{ij} + \tilde{\mu}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right).$$

The result of this calculation will be a linear combination of $\mathbf{U}(\mathbf{r}_0)$ and $\nabla_0^2 \mathbf{U}(\mathbf{r}_0)$, which will be substituted into (4.1), thus determining A and B.

Since the calculation is very lengthy, we have taken a number of notational shortcuts to organize the work. First, write the boundary condition at the sphere surface as

$$\mathbf{v} = -\mathbf{U}(\mathbf{r}_0 - a\hat{\mathbf{n}} + a\hat{\mathbf{r}}). \tag{4.10}$$

$\mathbf{r}_0$, $\hat{\mathbf{n}}$ and $\mathbf{r}_1 = \mathbf{r}_0 - a\hat{\mathbf{n}}$ are fixed here, while $\hat{\mathbf{r}}$ is the running normal to the sphere. This can be written as a symbolic Taylor expansion by

$$\begin{aligned}\mathbf{v} &= -\exp\{a(\hat{\mathbf{r}}-\hat{\mathbf{n}}).\nabla_0\}\,\mathbf{U}(\mathbf{r}_0)\\ &= -\exp\{-\hat{\mathbf{r}}.\mathbf{S}\}\,\mathbf{U}_1,\ \mathbf{U}_1 = \mathbf{U}(\mathbf{r}_1) = \exp\{\hat{\mathbf{n}}.\mathbf{S}\}\,\mathbf{U}(\mathbf{r}_0),\end{aligned} \tag{4.11}$$

where the operator $\mathbf{S} = a\nabla_0$ operates on the $\mathbf{r}_0$ dependence of $\mathbf{U}(\mathbf{r}_0)$. $\mathbf{S}$ will be treated as a constant until the very end, then it will be allowed to operate on $\mathbf{U}(\mathbf{r}_0)$. We note that, since $\operatorname{div}\mathbf{U} = 0$, $\mathbf{S}$ has the property $\mathbf{S}.\mathbf{U}_1 = 0$ and $\mathbf{S}.\mathbf{U}(\mathbf{r}_0) = 0$. This will be used repeatedly. Now, since $\mathbf{S}$ is to be treated as a constant, let it be the polar direction for spherical co-ordinates centred at the centre of the sphere. Take $\mathbf{S} = \hat{\mathbf{k}}S$, $\hat{\mathbf{k}}$ a unit vector, then the boundary condition becomes

$$\mathbf{v} = -\exp\{-S\cos\theta\}\,\mathbf{U}_1 \tag{4.12}$$

at $r = a$. r is the radial co-ordinate, θ the polar angle and β will be the circumferential angle. The problem has been changed from one with an arbitrary function in the boundary conditions to one with a specific function. This could alternatively be done by expressing $\mathbf{U}$ as a Fourier integral.

Following Morse & Feshbach (1953), we use the following representation:

$$\mathbf{v} = -\nabla\times\mathbf{r}\psi_1 - \nabla\times(\nabla\times\mathbf{r}\psi_2), \tag{4.13}$$

where the position vector $\mathbf{r}$ is now centred at the sphere centre, $\mathbf{r} = r\hat{\mathbf{r}}$. This automatically satisfies $\operatorname{div}\mathbf{v} = 0$ and will satisfy (4.8) if the scalars ψ_1 and ψ_2 are solutions of

$$\nabla^2\psi_1 = \alpha^2\psi_1, \quad \nabla^2\psi_2 = \psi_0, \quad \nabla^2\psi_0 = \alpha^2\psi_0, \tag{4.14}$$

ψ_0 being defined by the second equation. The pressure, given by

$$p = -\tilde{\mu}[\psi_0 - \alpha^2\psi_2 + \mathbf{r}.\nabla(\psi_0 - \alpha^2\psi_2)], \tag{4.15}$$

satisfies $\nabla^2 p = 0$ by virtue of (4.14). The functions ψ_0, ψ_1, ψ_2 are functions of r, θ, β but the β dependence may be separated out, eliminating the need for vector spherical harmonics, by noting that the solution must be linear in the arbitrary constant vector $\mathbf{U}_1$. This suggests the change of variables

$$\left.\begin{aligned}\psi_0 &= g_0(r,\theta)\,\mathbf{U}_1.\hat{\mathbf{r}},\\ \psi_2 &= g_2(r,\theta)\,\mathbf{U}_1.\hat{\mathbf{r}},\\ \psi_1 &= h_1(r,\theta)\,\mathbf{U}_1.\hat{\mathbf{r}}\times\hat{\mathbf{k}},\end{aligned}\right\} \tag{4.16}$$

selected, so that $\mathbf{v}$ will be a polar vector (Landau & Lifshitz 1959). The proof of this is that the resulting velocity field can be made to satisfy the boundary conditions. The functions g_0, g_1, g_2 must be solutions of

$$\left.\begin{aligned} \nabla^2 g_0 - \frac{2g_0}{r^2} + \frac{2\cos\theta}{\sin\theta}\frac{1}{r^2}\frac{\partial g_0}{\partial\theta} &= \alpha^2 g_0, \\ \nabla^2 g_2 - \frac{2g_2}{r^2} + \frac{2\cos\theta}{\sin\theta}\frac{1}{r^2}\frac{\partial g_2}{\partial\theta} &= g_0, \\ \nabla^2 g_1 - \frac{2g_1}{r^2} + \frac{2\cos\theta}{\sin\theta}\frac{1}{r^2}\frac{\partial g_1}{\partial\theta} &= \alpha^2 g_1. \end{aligned}\right\} \tag{4.17}$$

Substituting (4.16) into (4.13), and using (4.11), gives three boundary conditions inter-relating g_0, g_1, g_2. After re-arrangement, several of these may be integrated over the angle variable, and simplify to

$$(1-Z^2)\,g_1\big|_{r=a} = -\frac{1}{S}\exp\{SZ\} + \frac{1}{S}\cosh S + \frac{Z}{S}\sinh S, \tag{4.18}$$

$$(1-Z^2)\,g_2\big|_{r=a} = -a\left(\frac{1}{S}\exp\{SZ\} - \frac{1}{S^2}\cosh S - \frac{Z}{S^2}\sinh S\right), \tag{4.19}$$

$$(1-Z^2)\frac{\partial g_2}{\partial r}\bigg|_{r=a} = \left(\frac{1}{S^2} - \frac{Z}{S}\right)(\exp\{SZ\} - \cosh S - Z\sinh S) + (1-Z^2)\frac{\sinh S}{S}, \tag{4.20}$$

where $Z = \cos\theta$.

Solutions of (4.17) which tend to zero at infinity are

$$g_0 = \sum_{j=1}^{\infty} a_j H_j(\alpha r)\frac{dP_j(Z)}{dZ}, \tag{4.21}$$

$$g_1 = \sum_{j=1}^{\infty} b_j H_j(\alpha r)\frac{dP_j(Z)}{dZ}, \tag{4.22}$$

$$g_2 = \sum_{j=1}^{\infty}\left(\frac{a_j}{\alpha^2}H_j(\alpha r) + c_j r^{-j-1}\right)\frac{dP_j(Z)}{dZ}, \tag{4.23}$$

where the $P_j(Z)$ are Legendre polynomials, a_j, b_j, c_j are constants to be determined, and the $H_j(\alpha r)$ are modified spherical Bessel functions defined by

$$\begin{aligned} H_j(\zeta) &= -\left(\frac{2}{\pi\zeta}\right)^{\frac{1}{2}} K_{j+\frac{1}{2}}(\zeta) \\ &= -(-1)^j \zeta^j \left(\frac{1}{\zeta}\frac{d}{d\zeta}\right)^j \frac{\exp\{-\zeta\}}{\zeta}. \end{aligned} \tag{4.24}$$

These series are substituted into the boundary conditions, and use is made of the orthogonality of the Legendre functions and the integral

$$\tfrac{1}{2}\int_{-1}^{1} \exp\{SZ\}\,P_j(Z)\,dZ = G_j(S), \tag{4.25}$$

to determine a_j, b_j, c_j. The function G_j, defined above, may be expressed as

$$G_j(\zeta) = i^n j_j(-i\zeta) = \left(\frac{\pi}{2\zeta}\right)^{\frac{1}{2}} I_{j+\frac{1}{2}}(\zeta) = \zeta^j\left(\frac{1}{\zeta}\frac{d}{d\zeta}\right)\frac{\sinh\zeta}{\zeta}, \tag{4.26}$$

where j_j is a spherical Bessel function and $I_{j+\frac{1}{2}}$ is a modified Bessel function. The coefficients are found to be

$$b_j = \frac{2j+1}{j(j+1)}\frac{G_j(S)}{H_j(\alpha a)}, \tag{4.27}$$

$$a_j = -\frac{2j+1}{j(j+1)}\alpha\left((j+1)\frac{G_j(S)}{S}+G_j'(S)\right)\Big/H_{j-1}(\alpha a), \tag{4.28}$$

$$c_j = \frac{2j+1}{j(j+1)}\frac{a^{j+1}}{\alpha}\frac{H_j(\alpha a)}{H_{j-1}(\alpha a)}\left((j+1)\frac{G_j(S)}{S}+G_j'(S)\right)+\frac{2j+1}{j(j+1)}a^{j+2}\frac{G_j(S)}{S}. \tag{4.29}$$

The pressure may be expressed as

$$\left.\begin{aligned} p &= p_0\mathbf{U}_1.\mathbf{r}, \\ p_0 &= -\tilde{\mu}\sum_{j=1}^{\infty} j\alpha^2 c_j r^{-j-1}\frac{dP_j}{dZ}. \end{aligned}\right\} \tag{4.30}$$

Now the difficult part of the calculation remains. Substituting (4.13) into

$$\langle\mathbf{T}\rangle_1.\hat{\mathbf{r}} = -p\hat{\mathbf{r}}+\tilde{\mu}[\hat{\mathbf{r}}.\nabla\mathbf{v}+(\nabla\mathbf{v}).\hat{\mathbf{r}}] \tag{4.31}$$

gives

$$\begin{aligned}\langle\mathbf{T}\rangle_1.\hat{\mathbf{r}} = & -p_0\mathbf{U}_1.\hat{\mathbf{r}}\hat{\mathbf{r}}+\tilde{\mu}\left\{\hat{\mathbf{r}}\hat{\mathbf{r}}\times\nabla\left(\frac{\partial g_1}{\partial r}-\frac{g_1}{r}\right)\hat{\mathbf{r}}.\hat{\mathbf{k}}\times\mathbf{U}_1\right.\\ & \left.+\left(\frac{\partial g_1}{\partial r}-\frac{g_1}{r}\right)(\hat{\mathbf{r}}.\mathbf{U}_1\hat{\mathbf{k}}-\hat{\mathbf{r}}.\hat{\mathbf{k}}\mathbf{U}_1)\right\}+\tilde{\mu}\left\{\left(r\frac{\partial g_0}{\partial r}+2g_0\right)\mathbf{U}_1.\hat{\mathbf{r}}+(r\nabla g_0-\hat{\mathbf{r}}g_0)\,\mathbf{U}_1.\hat{\mathbf{r}}\right.\\ & \left.+rg_0\mathbf{U}_1-2\frac{\partial}{\partial r}\left(\frac{\partial g_2}{\partial r}+\frac{g_2}{r}\right)\mathbf{U}_1-2\mathbf{U}_1.\hat{\mathbf{r}}\nabla r\frac{\partial}{\partial r}\left(\frac{\partial g_2}{\partial r}+\frac{g_2}{r}\right)\right\}.\end{aligned} \tag{4.32}$$

On the surface of the sphere, this is of form

$$\langle\mathbf{T}\rangle_1.\hat{\mathbf{r}} = \mathbf{Q}(\theta,\beta).\mathbf{U}_1,$$

where $\mathbf{Q}$ is a dyadic. To evaluate

$$a^2\int_{\mathbf{r}_1=\mathbf{r}_0-a\hat{\mathbf{n}},\ \mathbf{r}_0\ \text{fixed}}\langle\mathbf{T}\rangle_1.\hat{\mathbf{n}}\,d\hat{\mathbf{n}}$$

we must set $\hat{\mathbf{r}}$, the running normal, equal to $\hat{\mathbf{n}}$, the normal at the point $\mathbf{r}_0$ when the sphere is at $\mathbf{r}_1$. Let θ_0, β_0 be spherical co-ordinates for $\hat{\mathbf{n}}$ and note from (4.11) that $\mathbf{U}_1 = \exp\{\hat{\mathbf{n}}.\mathbf{S}\}\,\mathbf{U}(\mathbf{r}_0) = \exp S\cos\theta\,\mathbf{U}(\mathbf{r}_0)$. Then the integral we want is

$$a^2\int\exp\{S\cos\theta_0\}\,\mathbf{Q}(\theta_0,\beta_0).\mathbf{U}(\mathbf{r}_0)\sin\theta_0\,d\theta_0\,d\beta_0. \tag{4.33}$$

Fortunately the integration on β_0 can be carried out while $\mathbf{Q}$ is in the form given by (4.32). Integrals like

$$\int\hat{\mathbf{n}}[\hat{\mathbf{n}}d\beta_0.\mathbf{U}(\mathbf{r}_0)] = \pi\sin^2\theta_0\,\mathbf{U}(\mathbf{r}_0),$$

which make use of $\mathbf{k}.\mathbf{U} = 0$, can be evaluated. There remains substitution of the series forms and integration over θ_0. The final result of this lengthy calculation is

$$a^2 \int_{\mathbf{r}_1 = \mathbf{r}_0 - a\hat{\mathbf{n}},\ \mathbf{r}_0 \text{ fixed}} \langle \mathbf{T} \rangle_1 . \hat{\mathbf{n}}\, d\hat{\mathbf{n}} = \tilde{\mu} a F(S^2, \alpha a)\, \mathbf{U}(\mathbf{r}_0), \tag{4.34}$$

where

$$F(S^2, \alpha a) = \pi \sum_{j=1}^{\infty} (-1)^j (2j+1) \left\{ -2j \frac{\alpha^2 a^2}{S^2} - 2(j+1) \right\} G_j(S)^2$$
$$+ 8\pi \sum_{j=1}^{\infty} (-1)^j (j+1) \frac{\alpha a\, H_{j+1}(\alpha a)}{H_j(\alpha a)} G_j(S)^2 + 6\pi \alpha a \frac{H_1(\alpha a)}{H_0(\alpha a)} G_0(S)^2. \tag{4.35}$$

Now we interpret F as an operator operating on $\mathbf{U}(\mathbf{r}_0)$. Since $\mathbf{S} = a\nabla_0$, we have $S^2 = a^2\nabla_0^2$. As a temporary notation F can be written as a series,

$$F(S^2, \alpha a) = \sum_{j=0}^{\infty} \lambda_j S^{2j} = \sum_{j=0}^{\infty} \lambda_j (a^2 \nabla_0^2)^j \tag{4.36}$$

which operates on $\mathbf{U}(\mathbf{r}_0)$ term by term:

$$F(S^2, \alpha a)\, \mathbf{U}(\mathbf{r}_0) = \lambda_0 \mathbf{U}(\mathbf{r}_0) + \lambda_1 \nabla_0^2 \mathbf{U}(\mathbf{r}_0) + \lambda_2 \nabla_0^4 \mathbf{U}(\mathbf{r}_0) + \dots. \tag{4.37}$$

Since $\mathbf{U}$ is a solution of (4.5) and $\nabla^2 p = 0$, we have

$$\left.\begin{aligned} \nabla_0^4 \mathbf{U}(\mathbf{r}_0) &= \alpha^2 \nabla_0^2 \mathbf{U}(\mathbf{r}_0), \\ \nabla_0^6 \mathbf{U}(\mathbf{r}_0) &= \alpha^4 \nabla_0^2 \mathbf{U}(\mathbf{r}_0), \end{aligned}\right\} \tag{4.38}$$

and so on. This is an essential result, for then

$$\begin{aligned} F(S^2, \alpha a)\, \mathbf{U} &= \lambda_0 \mathbf{U} + \sum_{j=1}^{\infty} \lambda_j \frac{(a^2\alpha^2)^j}{\alpha^2} \nabla_0^2 \mathbf{U} \\ &= F(0, \alpha a)\, \mathbf{U} + \frac{F(\alpha^2 a^2, \alpha a) - F(0, \alpha a)}{\alpha^2} \nabla_0^2 \mathbf{U}. \end{aligned} \tag{4.39}$$

Therefore using (4.34) and (4.1) we have

$$\begin{aligned} \mathscr{F}(\langle \mathbf{u} \rangle) &= \tilde{\mu} n a F(0, \alpha a) \langle \mathbf{u} \rangle + \frac{\tilde{\mu} n a}{\alpha^2} (F(\alpha^2 a^2, \alpha a) - F(0, \alpha a)) \nabla^2 \langle \mathbf{u} \rangle \\ &= A \langle \mathbf{u} \rangle + B \nabla^2 \langle \mathbf{u} \rangle \end{aligned} \tag{4.40}$$

as was to be shown. Further, equating coefficients of $\langle \mathbf{u} \rangle$ and $\nabla^2 \langle \mathbf{u} \rangle$, and using (4.3) and (4.4) for A and B, gives

$$\tilde{\mu} n a F(0, \alpha a) = \tilde{\mu}(1-\phi)\alpha^2,$$
$$\frac{\tilde{\mu} n a}{\alpha^2} (F(\alpha^2 a^2, \alpha a) - F(0, \alpha a)) = \mu - (1-\phi)\, \tilde{\mu},$$

or simplifying, using $\tilde{\mu}/\mu = M$ and eliminating n by using $\phi = 4\pi a^3 n/3$, we get

$$\frac{3\phi}{4\pi} F(0, \alpha a) = (1-\phi)\, \alpha^2 a^2, \tag{4.41}$$

$$\frac{3\phi}{4\pi} M F(\alpha^2 a^2, \alpha a) = \alpha^2 a^2, \tag{4.42}$$

two equations to determine M and αa as functions of ϕ, the volume fraction of solids.

When $S = 0$, the series defining F terminates at one term, and we find

$$F(0, \alpha a) = 6\pi(1+\alpha a+\tfrac{1}{9}\alpha^2 a^2). \tag{4.43}$$

When $S^2 = \alpha^2 a^2$ we can also simplify somewhat, since the arguments of the H_j and G_j functions are the same. With the identity $G_j H_{j+1} + G_{j+1} H_j = -1/\alpha^2 a^2$ we get

$$F(\alpha^2 a^2, \alpha a) = -2\pi \sum_{j=1}^{\infty} (-1)^j (2j+1)^2 G_j(\alpha a)^2 - 8\pi \sum_{j=1}^{\infty} (-1)^j (j+1) \times \left(\alpha a G_j(\alpha a)\, G_{j+1}(\alpha a) + \frac{G_j(\alpha a)}{\alpha a H_j(\alpha a)}\right) + 6\pi(1+\alpha a)\left(\frac{\sinh \alpha a}{\alpha a}\right)^2. \tag{4.44}$$

With $F(0, \alpha a)$ given above, (4.41) gives the same quadratic equation for αa that obtained by Brinkman, (3.6), so αa is given by

$$\alpha a = \frac{\frac{9}{4}\phi + \frac{3}{4}(8\phi - 3\phi^2)^{\frac{1}{2}}}{1-\frac{3}{2}\phi}. \tag{4.45}$$

Also, the permeability is given by (3.7):

$$\frac{k}{k_0} = \frac{1}{M(\phi)}(1+\tfrac{3}{4}[\phi - (8\phi - 3\phi^2)^{\frac{1}{2}}]), \tag{4.46}$$

with the effective viscosity $M(\phi)$ obtained from (4.42),

$$M(\phi) = \frac{4\pi}{3} \frac{a^2\alpha^2}{\phi F(\alpha^2 a^2, \alpha a)}, \tag{4.47}$$

αa being a function of ϕ through (4.45).

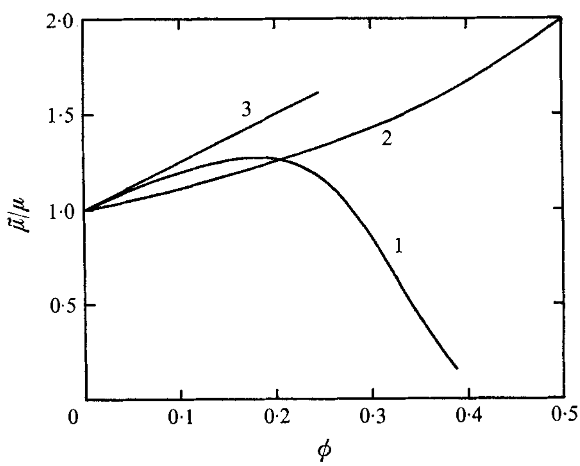

FIGURE 2. Effective viscosity $\tilde{\mu}/\mu$ in a random bed of fixed spheres versus the volume fraction of spheres ϕ. 1, calculated from (4.47); 2, calculated from $1/(1-\phi)$; 3, calculated from the Einstein result $1+2{\cdot}5\phi$.

These functions have been computed. The function $M(\phi)$ is shown in figure 2. As suspected by Brinkman, it is not always greater than one, in fact it decreases rapidly when $\phi > 0{\cdot}3$. This is a source of difficulty, since the resulting permeability shown in figure 1 follows the Brinkman result fairly closely, and then suddenly diverges from it where the viscosity becomes small. It is felt that this occurs because of the approximations made. Neglecting the inpenetrability of the particles is an approximation which becomes worse as the particles become more crowded. It is not surprising that there is an effect at a volume concentration of 30%, since the average distance between particle centres is only 1·25 diameters. In order to get a valid result at higher concentrations, it would be necessary to account for the void sphere around each particle.

The significant result of this section is the theoretical justification for the equations

$$\operatorname{div}\langle\mathbf{u}\rangle = 0,$$
$$0 = -\nabla\bar{p} + \tilde{\mu}\nabla^2\langle\mathbf{u}\rangle + \mu/k\langle\mathbf{u}\rangle,$$

describing the flow through porous media. It was pointed out by Tam (1969) that, whenever the spatial length scale is much greater than $1/\alpha$ (α defined by $\mu/k = \tilde{\mu}\alpha^2$), the $\nabla^2\langle\mathbf{u}\rangle$ term is negligible. For large systems this means that Darcy's law

$$0 = -\nabla\bar{p} + \mu/k\langle\mathbf{u}\rangle$$

is valid outside of boundary layers of thickness $1/\alpha$.

5. The viscosity and settling velocity of a suspension of spheres

The method used in §4 accounts for the effective viscosity in the flow through a bed of fixed spheres, which suggests using the same method to determine the effective viscosity of a suspension of spheres. This problem is more interesting because of the classical result of Einstein (1906, 1911) for dilute suspensions,

$$M = 1 + 2{\cdot}5\phi + \dots.$$

We shall present a generalization for moderately large ϕ.

As in the previous sections, we have an ensemble of systems of spheres with centres at $\mathbf{r}_i$, $i = 1, 2, \dots, N$. Let $\mathbf{U}_i$ be the velocity of the centre of the ith sphere and $\mathbf{\Omega}_i$ its angular velocity. The velocity of any point $\mathbf{r}$ in the ith sphere is $\mathbf{U}_i + \mathbf{\Omega}_i \times (\mathbf{r} - \mathbf{r}_i)$. The quantities $\mathbf{U}_i$ and $\mathbf{\Omega}_i$ are functionals of the positions of the other particles determined by the conditions that the torque produced by the hydrodynamic tractions be zero and the force produced by the tractions be balanced by the weight of the particle $\frac{4}{3}\pi a^3\rho_s\mathbf{g}$, where ρ_s is the mass density of the solid particle

As before, we assume that the ensemble average velocity satisfies

$$\left.\begin{aligned} \operatorname{div}\langle\mathbf{u}\rangle &= 0,\\ 0 &= -\nabla(\bar{p} - \rho\mathbf{g}\,.\,\mathbf{r}) + \frac{\mu}{1-\phi}\nabla^2\langle\mathbf{u}\rangle - \frac{\mathscr{F}(\langle\mathbf{u}\rangle)}{1-\phi},\end{aligned}\right\} \tag{5.1}$$

and the sub-ensemble average velocity with particle 1 fixed satisfies the same equations. Let $\mathbf{V}(\mathbf{r}_1)$ and $\mathbf{\Omega}(\mathbf{r}_1)$ be the *average* velocity and *average* angular velocity of particle 1. They are calculated from force and torque balances using the average traction $\langle \mathbf{T}\rangle_1 . \hat{\mathbf{r}}$. The boundary condition $\langle \mathbf{u}\rangle_1 = \mathbf{V} + a\hat{\mathbf{r}} \times \mathbf{\Omega}$ on $\mathbf{r} = \mathbf{r}_1 + a\hat{\mathbf{r}}$ is obtained by averaging with the particle fixed. The complete statement of the problem for $\langle \mathbf{u}_1\rangle$ is

$$\left.\begin{aligned} \operatorname{div}\langle \mathbf{u}\rangle_1 &= 0, \\ 0 &= -\nabla(\overline{p}_1 - \rho \mathbf{g} . \hat{\mathbf{r}}) + \frac{\mu}{1-\phi}\nabla^2\langle \mathbf{u}_1\rangle - \frac{\mathscr{F}(\langle \mathbf{u}\rangle_1)}{1-\phi}, \end{aligned}\right\} \tag{5.2}$$

to be solved with boundary conditions

$$\left.\begin{aligned} &\langle \mathbf{u}\rangle_1 = \mathbf{V} + a\hat{\mathbf{r}} \times \mathbf{\Omega} \quad \text{on} \quad \mathbf{r} = \mathbf{r}_1 + a\hat{\mathbf{r}}, \\ &\langle \mathbf{u}\rangle_1 \to \langle \mathbf{u}\rangle \quad \text{at infinity}, \end{aligned}\right\} \tag{5.3}$$

with $\mathbf{V}$ and $\mathbf{\Omega}$ related to $\langle \mathbf{u}_1\rangle$ by the two conditions

$$a^3 \int_{\mathbf{r}_1 \text{ fixed}, \, \mathbf{r} = \mathbf{r}_1 + a\hat{\mathbf{r}}} \hat{\mathbf{r}} \times \langle \mathbf{T}\rangle_1 . \hat{\mathbf{r}} \, d\hat{\mathbf{r}} = 0, \tag{5.4}$$

$$a^2 \int_{\mathbf{r}_1 \text{ fixed}, \, \mathbf{r} = \mathbf{r}_1 + a\hat{\mathbf{r}}} \langle \mathbf{T}\rangle_1 . \hat{\mathbf{r}} \, d\hat{\mathbf{r}} + \rho_s \tfrac{4}{3}\pi a^3 \mathbf{g} = 0, \tag{5.5}$$

and the functional $\mathscr{F}(\langle \mathbf{u}\rangle)$ is related by

$$\mathscr{F}(\langle \mathbf{u}\rangle) = na^2 \int_{\mathbf{r}_1 = \mathbf{r}_0 - a\hat{\mathbf{n}}, \, \mathbf{r}_0 \text{ fixed}} \langle \mathbf{T}\rangle_1 . \hat{\mathbf{n}} \, d\hat{\mathbf{n}}, \tag{5.6}$$

where
$$\langle T_{ij}\rangle_1 = -\overline{p}_1 \delta_{ij} + \tilde{\mu}\left(\frac{\partial \langle u_i\rangle_1}{\partial x_j} + \frac{\partial \langle u_j\rangle_1}{\partial x_i}\right).$$

The key to solving this problem, as with the stationary sphere problem, is to anticipate what form $\mathscr{F}(\langle \mathbf{u}\rangle)$ should have. We might try $A(\langle \mathbf{u}\rangle - \mathbf{V}) + B\nabla^2\langle \mathbf{u}\rangle$ (i.e. a resistance proportional to the relative particle velocity), but this relative velocity turns out to be the sum of terms proportional to $\nabla^2\langle \mathbf{u}\rangle$ and to $\mathbf{g}$. Anticipating this, we assume

$$\mathscr{F}(\langle \mathbf{u}\rangle) = B\nabla^2\langle \mathbf{u}\rangle + C\mathbf{g}. \tag{5.7}$$

Then $\langle \mathbf{u}\rangle_1$ must satisfy

$$0 = -\nabla\left(\overline{p}_1 - \rho \mathbf{g} . \mathbf{r} + \frac{C\mathbf{g} . \mathbf{r}}{1-\phi}\right) + \tilde{\mu}\nabla^2\langle \mathbf{u}\rangle_1, \tag{5.8}$$

where
$$\tilde{\mu} = \frac{\mu - B}{1-\phi} \tag{5.9}$$

is the effective viscosity. This is just Stokes's equation, so the problem is actually a little easier than for stationary spheres.

Let
$$\left.\begin{aligned} \langle \mathbf{u}\rangle &= \mathbf{U}, \\ \overline{p}_0 &= \overline{p} - \rho \mathbf{g} . \mathbf{r} + C\mathbf{g} . \mathbf{r}/(1-\phi) \end{aligned}\right\} \tag{5.10}$$

be the unperturbed velocity and pressure. Let

$$\left.\begin{aligned} \langle \mathbf{u}\rangle_1 &= \mathbf{U}+\mathbf{\upsilon}, \\ \bar{p}_1-\rho\mathbf{g}.\mathbf{r}+C\mathbf{g}.\mathbf{r}/(1-\phi) &= \bar{p}_0+p. \end{aligned}\right\} \tag{5.11}$$

Then

$$\left.\begin{aligned} \operatorname{div}\mathbf{\upsilon} &= 0, \\ 0 &= -\nabla p+\tilde{\mu}\nabla^2\mathbf{\upsilon}, \end{aligned}\right\} \tag{5.12}$$

with boundary conditions

$$\left.\begin{aligned} \mathbf{\upsilon} &\to 0 \quad \text{at infinity}, \\ \mathbf{\upsilon} &= \mathbf{V}(\mathbf{r}_1)+a\hat{\mathbf{r}}\times\mathbf{\Omega}(\mathbf{r}_1)-\mathbf{U}(\mathbf{r}_1+a\hat{\mathbf{r}}) \quad \text{on the sphere}. \end{aligned}\right\} \tag{5.13}$$

With the solution we need to calculate the integrals in (5.4), (5.5) and (5.6).

We will solve this problem as the superposition of the three problems (i), (ii), (iii), below, where the boundary condition on the sphere consists of each of the three terms in (5.13) separately.

(i) The case where the boundary conditions are

$$\mathbf{\upsilon}\to 0 \quad \text{as} \quad \mathbf{r}\to\infty,$$

$$\mathbf{\upsilon} = \mathbf{V}(\mathbf{r}_1) \text{ (a constant)} \quad \text{on} \quad \mathbf{r} = \mathbf{r}_1+a\hat{\mathbf{r}},$$

is given by Landau & Lifshitz (1959) as

$$\mathbf{\upsilon} = \nabla\times(\nabla\times f(r)\,\mathbf{V}), \tag{5.14}$$

with

$$f = -\tfrac{3}{4}ar-\tfrac{1}{4}\frac{a^3}{r}, \tag{5.15}$$

and

$$p = \frac{3}{2}\frac{a}{r^2}\tilde{\mu}\mathbf{V}.\mathbf{r}. \tag{5.16}$$

The traction on the surface due to this part of the perturbation field is

$$\langle \mathbf{T}\rangle_1.\hat{\mathbf{r}} = -\frac{3\tilde{\mu}}{2a}\mathbf{V}. \tag{5.17}$$

We calculate the contribution of this traction to each of the three integrals we need. The contribution to the torque integral (5.4) is clearly zero. The contribution to the force (5.5) is

$$a^2\int_{\mathbf{r}_1\text{ fixed},\ \mathbf{r}=\mathbf{r}_1+a\hat{\mathbf{r}}}\langle \mathbf{T}\rangle_1.\hat{\mathbf{r}}\,d\hat{\mathbf{r}} = -6\pi\tilde{\mu}a\mathbf{V}(\mathbf{r}_1). \tag{5.18}$$

The contribution to the stress at a point in (5.6) is more difficult, since $\mathbf{r}_1$ is not fixed in this integral, therefore $\mathbf{V}(\mathbf{r}_1)$ will vary. We write

$$\mathbf{V}(\mathbf{r}_1) = \mathbf{V}(\mathbf{r}_0-a\hat{\mathbf{n}}) = \exp\{-\mathbf{S}.\hat{\mathbf{n}}\}\mathbf{V}(\mathbf{r}_0) \quad \text{with} \quad \mathbf{S} = a\nabla_0$$

as before. Then

$$\begin{aligned} a^2\int_{\mathbf{r}_1=\mathbf{r}_0-a\hat{\mathbf{n}},\ \mathbf{r}_0\text{ fixed}}\langle \mathbf{T}\rangle_1.\hat{\mathbf{n}}\,d\hat{\mathbf{n}} &= a^2\int\exp\{-\mathbf{S}.\hat{\mathbf{n}}\}\left\{-\frac{3}{2}\frac{\tilde{\mu}}{a}\mathbf{V}(\mathbf{r}_0)\right\}d\hat{\mathbf{n}} \\ &= -6\pi\tilde{\mu}a\frac{\sinh S}{S}\mathbf{V}(\mathbf{r}_0). \end{aligned} \tag{5.19}$$

It will be shown later that $\nabla^4\mathbf{V} = 0$, therefore, since $S^2 = a^2\nabla_0^2$, we need only expand the above function of S^2 in power series up to terms in S^2, all higher powers giving zero when they operate on $\mathbf{V}(\mathbf{r}_0)$. We get

$$a^2\int_{\mathbf{r}_1=\mathbf{r}_0-a\hat{\mathbf{n}},\ \mathbf{r}_0\text{ fixed}} \langle\mathbf{T}\rangle_1.\hat{\mathbf{n}}\,d\hat{\mathbf{n}} = -6\pi\tilde{\mu}a\mathbf{V}(\mathbf{r}_0)-\pi\tilde{\mu}a^3\nabla_0^2\mathbf{V}(\mathbf{r}_0). \tag{5.20}$$

(ii) The solution to (5.12), with boundary conditions

$$\mathbf{\upsilon}\to 0 \quad \text{as} \quad \mathbf{r}\to\infty,$$
$$\mathbf{\upsilon} = a\hat{\mathbf{r}}\times\mathbf{\Omega}(\mathbf{r}_1) \quad \text{on} \quad \mathbf{r} = \mathbf{r}_1+a\hat{\mathbf{r}},$$

is also given by Landau & Lifshitz (1959). It is simply

$$\mathbf{\upsilon} = \frac{a^3}{r^2}\mathbf{\Omega}(\mathbf{r}_1)\times\hat{\mathbf{r}}, \tag{5.21}$$

and the pressure is zero. The traction at a point on the sphere is

$$\langle\mathbf{T}\rangle_1.\hat{\mathbf{r}} = -3\tilde{\mu}\mathbf{\Omega}(\mathbf{r}_1)\times\hat{\mathbf{r}}.$$

The contribution to the torque integral (5.4) is

$$a^3\int_{\mathbf{r}_1\text{ fixed},\ \mathbf{r}=\mathbf{r}_1+a\hat{\mathbf{r}}} \hat{\mathbf{r}}\times\langle\mathbf{T}\rangle_1.\hat{\mathbf{r}}\,d\hat{\mathbf{r}} = -8\pi\tilde{\mu}a^3\mathbf{\Omega}(\mathbf{r}_1). \tag{5.22}$$

The contribution to the force (5.5) is zero, but there is a contribution to the mean traction at a point. Writing $\mathbf{\Omega}(\mathbf{r}_1) = \mathbf{\Omega}(\mathbf{r}_0-a\hat{\mathbf{n}}) = \exp\{-\mathbf{S}.\hat{\mathbf{n}}\}\mathbf{\Omega}(\mathbf{r}_0)$ we have

$$a^2\int_{\mathbf{r}_1=\mathbf{r}_0-a\hat{\mathbf{n}},\ \mathbf{r}_0\text{ fixed}} \langle\mathbf{T}\rangle_1.\mathbf{n}\,d\mathbf{n} = -3\tilde{\mu}a^2\int\exp\{-\mathbf{S}.\hat{\mathbf{n}}\}\mathbf{\Omega}(\mathbf{r}_0)\times\hat{\mathbf{n}}\,d\hat{\mathbf{n}}$$
$$= -12\tilde{\mu}a^2\frac{1}{S}\frac{d}{dS}\frac{\sinh S}{S}\mathbf{S}\times\mathbf{\Omega}(\mathbf{r}_0). \tag{5.23}$$

Again we shall use a result which will be obtained later. That is $\mathbf{\Omega} = \frac{1}{2}\nabla\times\mathbf{U}$. Then $\mathbf{S}\times\mathbf{\Omega}(\mathbf{r}_1) = \frac{1}{2}a\nabla_0\times(\nabla_0\times\mathbf{U}(\mathbf{r}_0)) = -\frac{1}{2}a\nabla_0^2\mathbf{U}(\mathbf{r}_0)$ since $\operatorname{div}\mathbf{U} = 0$. Now, since $\nabla^4\mathbf{U} = 0$, a power series expansion in S^2 gives

$$a^2\int_{\mathbf{r}_1=\mathbf{r}_0-a\hat{\mathbf{n}},\ \mathbf{r}_0\text{ fixed}} \langle\mathbf{T}\rangle_1.\hat{\mathbf{n}}\,d\hat{\mathbf{n}} = 2\pi\tilde{\mu}a^3\nabla_0^2\mathbf{U}(\mathbf{r}_0). \tag{5.24}$$

(iii) The problem with boundary conditions

$$\mathbf{\upsilon}\to 0 \quad \text{as} \quad \mathbf{r}\to\infty,$$
$$\mathbf{\upsilon} = -\mathbf{U}(\mathbf{r}_1+a\hat{\mathbf{r}}) \quad \text{on} \quad \mathbf{r} = \mathbf{r}_1+a\hat{\mathbf{r}}$$

is, of course, much more difficult than the above two problems. Much of the analysis with stationary spheres of § 4 can be adapted to this problem by putting $\alpha = 0$ there. The contribution to the torque integral (5.4) is

$$a^3\int_{\mathbf{r}_1\text{ fixed},\ \mathbf{r}=\mathbf{r}_1+a\hat{\mathbf{r}}} \hat{\mathbf{r}}\times\langle\mathbf{T}\rangle_1.\hat{\mathbf{r}}\,d\hat{\mathbf{r}} = 12\pi\tilde{\mu}a^2\frac{1}{S_1}\frac{d}{dS_1}\left(\frac{\sinh S_1}{S_1}\right)\mathbf{S}_1\times\mathbf{U}(\mathbf{r}_1), \tag{5.25}$$

where $\mathbf{S}_1 = a\nabla_1$ comes in from $\mathbf{U}(\mathbf{r}_1 + a\hat{\mathbf{r}}_1) = \exp\{\mathbf{S}_1 . \hat{\mathbf{r}}\}\,\mathbf{U}(\mathbf{r}_1)$. When the function of S_1 is expanded, interpreted as an operator, and $\nabla^2 (\nabla \times \mathbf{U}) = 0$ is used, we find

$$a^3 \int_{\mathbf{r}_1 \text{ fixed, } \mathbf{r}=\mathbf{r}_1+a\hat{\mathbf{r}}} \hat{\mathbf{r}} \times \langle \mathbf{T} \rangle_1 . \hat{\mathbf{r}}\, d\hat{\mathbf{r}} = 4\pi\tilde{\mu}a^3 \nabla_1 \times \mathbf{U}(\mathbf{r}_1). \tag{5.26}$$

Similarly, the contribution to the force on the particle is

$$a^2 \int_{\mathbf{r}_1 \text{ fixed, } \mathbf{r}=\mathbf{r}_1+a\hat{\mathbf{r}}} \langle \mathbf{T} \rangle_1 . \hat{\mathbf{r}}\, d\hat{\mathbf{r}} = 6\pi\tilde{\mu}a \left(\frac{2}{S_1}\frac{d}{dS_1} + \frac{d^2}{dS_1^2}\right)\frac{\sinh S_1}{S_1}$$
$$= 6\pi\tilde{\mu}a(\mathbf{U}(\mathbf{r}_1) + \tfrac{1}{6}a^2\nabla_1^2\mathbf{U}(\mathbf{r}_1)). \tag{5.27}$$

The mean traction at a point can be obtained from (4.34) by setting $\alpha = 0$. It has also been obtained by an independent calculation, by which means it is easier to obtain than (4.34). (This serves as a check on both results.) The result is

$$a^2 \int_{\mathbf{r}_1=\mathbf{r}_0-a\hat{\mathbf{n}},\ \mathbf{r}_0 \text{ fixed}} \langle \mathbf{T} \rangle_1 . \hat{\mathbf{n}}\, d\hat{\mathbf{n}} = \tilde{\mu}aF(S^2, 0)\,\mathbf{U}(\mathbf{r}_0)$$
$$= 6\pi\tilde{\mu}a\mathbf{U}(\mathbf{r}_0) - 2\pi\tilde{\mu}a^3\nabla_0^2\mathbf{U}(\mathbf{r}_0). \tag{5.28}$$

The last line comes from the first two terms in the power series expansion of $F(S^2, 0)$, since higher powers operating on $\mathbf{U}(\mathbf{r}_0)$ give zero.

In order to put all these results together, we need, additionally, the contributions of the unperturbed stress and gravitational terms to the torque and force, the contribution to the mean traction at a point being zero. The torque due to the unperturbed stress is zero by conservation of angular momentum, therefore we can substitute the contributions of (5.22), and (5.26) into (5.4), getting

$$\mathbf{\Omega} = \tfrac{1}{2}\nabla \times \mathbf{U}, \tag{5.29}$$

the result that was needed in (ii) to complete (5.24). This result, and (5.31) below, may be deduced from Faxen's formulae (Happel & Brenner 1965).

The contribution of the unperturbed stress and gravity to the force on the particle is found to be

$$-\left(\rho - \frac{C}{1-\phi}\right)\tfrac{4}{3}\pi a^3\mathbf{g}, \tag{5.30}$$

this being an effective buoyancy. A second term resulting from the unperturbed stress is zero by conservation of momentum. When this term is added to (5.18) and (5.27), and the weight of the particle is as indicated in (5.5), the final result is

$$\left(\rho_s - \rho + \frac{C}{1-\phi}\right)\tfrac{4}{3}\pi a^3\mathbf{g} - 6\pi\tilde{\mu}a\,(\mathbf{V} - \mathbf{U}) + \pi\tilde{\mu}a^3\nabla^2\mathbf{U} = 0. \tag{5.31}$$

This formula gives the particle velocity $\mathbf{V}$ in terms of $\mathbf{U}$. Since $\operatorname{div}\mathbf{U} = 0$, it follows that $\operatorname{div}\mathbf{V} = 0$. Also, since $\nabla^4\mathbf{U} = 0$, it follows that $\nabla^4\mathbf{V} = 0$, a result needed to get (5.20).

The mean traction at a point may now be found by adding the results of problems (i), (ii) and (iii), (5.20), (5.24), (5.28), after using (5.31) to eliminate $\mathbf{V}$. It is

$$a^2 \int_{\mathbf{r}_1=\mathbf{r}_0-a\hat{\mathbf{n}},\ \mathbf{r}_0 \text{ fixed}} \langle \mathbf{T} \rangle_1 . \hat{\mathbf{n}}\, d\hat{\mathbf{n}} = -\left(\rho_s - \rho + \frac{C}{1-\phi}\right)\tfrac{4}{3}\pi a^3\,\mathbf{g} - 2\pi\tilde{\mu}a^3\nabla_0^2\mathbf{U}(\mathbf{r}_0). \tag{5.32}$$

Therefore, using (5.6) and (5.7), we have

$$\mathscr{F}(\langle \mathbf{u} \rangle) = -n\left(\rho_s - \rho + \frac{C}{1-\phi}\right)\tfrac{4}{3}\pi a^3 \mathbf{g} - 2\pi\tilde{\mu}a^3 n \nabla^2 \langle \mathbf{u} \rangle$$

$$= C\mathbf{g} + B\nabla^2\langle \mathbf{u} \rangle, \tag{5.33}$$

as anticipated. With $B = \mu - (1-\phi)\tilde{\mu}, \phi = \frac{4}{3}\pi a^3 n$, we equate coefficients of $\mathbf{g}$ and $\nabla^2\langle \mathbf{u} \rangle$, finding

$$C = -\phi(1-\phi)(\rho_s - \rho) \tag{5.34}$$

and

$$\frac{\tilde{\mu}}{\mu} = \frac{1}{1-\frac{5}{2}\phi}. \tag{5.35}$$

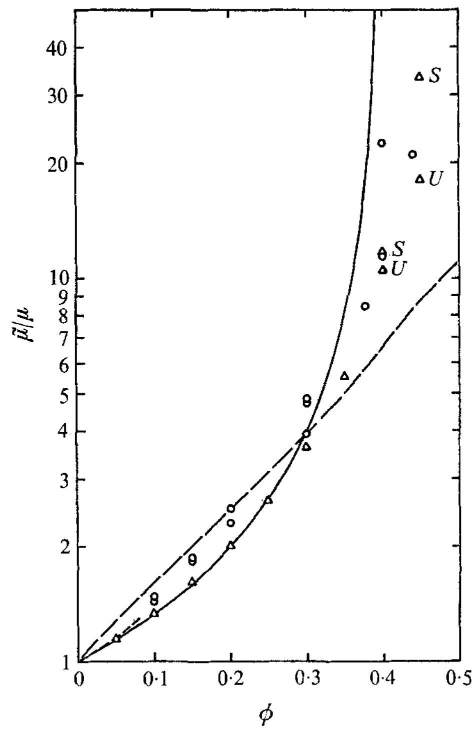

FIGURE 3. Effective viscosity $\tilde{\mu}/\mu$ of a suspension of spheres *vs.* the volume fraction of spheres ϕ. ——, calculated from (5.35); — —, calculated from Happel's (1957) free-surface theory. Δ, experiments of Vand (1948), (*S* stirred, *U* unstirred); ○, experiments of Williams (1953); ----, experiments of Cheng & Schachman (1955).

The latter result for the effective viscosity is really quite remarkable.† For small ϕ it gives $\tilde{\mu}/\mu = 1 + \frac{5}{2}\phi \ldots$, Einstein's result. Our expression becomes unbounded at $\phi = 0{\cdot}4$, which is clearly incorrect. However, when compared with measurements of Vand (1948), Williams (1953) and Cheng & Schachman (1955) in figure 3, it is seen that the agreement is quite good for $\phi < 0{\cdot}35$. In fact, this is about the limit for which one could expect the composite to behave like a Newtonian

† Budiansky (1965) has derived an expression for the shear modulus of a composite solid, which is equivalent to (5.35) by a 'self-consistent' analysis similar to Brinkman's. The exact analogy between Stokes flow and linear elasticity is described by Hashin (1970).

fluid, since Vand notes a difference due to stirring the mixture at $\phi = 0{\cdot}4$, as if particle contact were playing a role. Also, both Vand and Williams note viscosity ratios in excess of 200 at $\phi = 0{\cdot}5$, though not consistently. Equation (5.35) has been proposed by Ford (1960) on empirical grounds, comparing with the data of Vand and others. (A similar formula was proposed much earlier by Hess 1920.) Included in figure 3 is Happel's (1957) free-surface theory.

With C given by (5.34), (5.1) for the unperturbed velocity become

$$\left.\begin{aligned} \operatorname{div}\langle \mathbf{u}\rangle &= 0, \\ 0 &= -\nabla\bar{p}+\tilde{\mu}\nabla^2\langle \mathbf{u}\rangle+\bar{\rho}\mathbf{g}, \end{aligned}\right\} \tag{5.36}$$

where $\bar{\rho} = \phi\rho_s+(1-\phi)\rho$ is the mean density of the composite material. The last term is clearly a proper gravitational body force for the composite. From this equation we can show the consistency of using the effective viscosity when calculating stress at a solid boundary. Consider Poiseuille flow in a vertical pipe of radius R with a pure gravitational driving term. Integrating (5.36) over the cross-section of the pipe, with $\bar{p}$ constant, gives

$$\bar{\rho}g\pi R^3 = -2\pi R\left(\tilde{\mu}\frac{\partial u_z}{\partial r}\right). \tag{5.37}$$

That is, the weight of the fluid is balanced by a wall shear stress with an effective viscosity $\tilde{\mu}$.

Equation (5.31) for the particle velocity is also of some interest. Using C from (5.34), this can be written

$$(\rho_s-\bar{\rho})\tfrac{4}{3}\pi a^3\mathbf{g}-6\pi\tilde{\mu}a(\mathbf{V}-\langle\mathbf{u}\rangle)+\pi\tilde{\mu}a^3\nabla^2\langle\mathbf{u}\rangle = 0. \tag{5.38}$$

The first term is the weight of the particle minus its buoyancy in the composite material, the second is the Stokes drag, with the composite viscosity, as it moves relative to the composite velocity. The third is the drag due to the non-uniform velocity field. Since the motion of the composite material is determined independently from (5.36), this gives the settling velocity $\mathbf{V}$ after u has been found. Since $\operatorname{div}\mathbf{V} = 0$ there is no tendency for particles to accumulate anywhere. Uniform number density will persist.

For zero composite flow the settling velocity is

$$V = \frac{(\rho_s-\bar{\rho})\tfrac{4}{3}\pi a^3 g}{6\pi\tilde{\mu}a}, \tag{5.39}$$

a result which was derived by Kynch (1959) on intuitive grounds. Noting that $\rho_s-\bar{\rho} = (1-\phi)(\rho_s-\rho)$, and using $\tilde{\mu} = \mu/(1-\tfrac{5}{2}\phi)$, this can be presented as

$$V/V_0 = (1-\phi)(1-\tfrac{5}{2}\phi), \tag{5.40}$$

where V_0 is the settling velocity when $\phi = 0$.

The settling velocity is sometimes estimated from the flow through a stationary bed of spheres, using Darcy's equation for the flow resistance. Noting that the resistance per unit volume in Darcy's equation $(1-\phi)\,\mu V/k$ (V the relative velocity) equals the weight of the particles minus their buoyancy,

$$(1-\phi)\,(\rho_s-\rho)\,\tfrac{4}{3}\pi a^3 ng,$$

we get

$$V/V_0 = k/k_0, \tag{5.41}$$

k_0 being the permeability for the dilute case.† We have plotted V/V_0 from (5.40) in figure 1, in order to compare it with k/k_0 for stationary spheres. We have included in this figure the pertinent experimental results for settling velocity. The results of Richardson & Zaki (1954) for low Reynolds number sedimentation of glass spheres are fitted by $V/V_0 = (1-\phi)^{4\cdot 65}$. This fits our result fairly well. Also, the measurements of Cheng & Schachman (1955) for ultracentrifuge settling of extremely small polystyrene latex particles for small ϕ agrees well with our expression. Obviously there is considerable difference between the settling velocity result given by (5.40) and that given by the modified Brinkman result. The settling velocity for the suspension being larger than for the fixed bed of spheres.

6. Discussion

It has frequently been noted in the literature (citations in Davidson & Harrison 1963) that a fluid experiences a lower resistance in flow through a fluidized bed than through a fixed bed of similar material and the same porosity. That is, V/V_0 is larger for the fluidized bed, as we have found. The difference has been ascribed to various effects, such as slow internal circulation of the particles in the bed, tunnelling, formation of bubbles, aggregation. While all of these probably contribute to the phenomenon in practice, we see an appreciable difference in the results of the present analysis without any such effects. Since we have flow through two systems of randomly distributed particles, one fixed the other free, both analysed with similar approximations, it would appear that it is merely the added constraint in the fixed bed which increases the resistance.

A pair of particles settling side by side, each free to rotate about its centre, will settle faster than a pair which are prevented from rotating. Similarly, the drag on a sphere among randomly located spheres will be less if the sphere is free to rotate and adjust to the flow fields of the other particles. In the above reasoning, it is important that the spheres be randomly placed, or at least irregularly placed, for there would be no tendency to rotate in a regular cubic array. In our analysis of the motion of a single particle, the effects of other particles are averaged and only felt as part of the response of the composite fluid, so the direct effects described above are masked. Nevertheless, it is clear that, if one repeated the analysis of § 5 with the particle free to settle under gravity but constrained so that the angular velocity $\mathbf{\Omega}$ is zero, the effective viscosity would be greater and the settling velocity smaller. In fact, it is easy to check through the calculation of § 5 with $\mathbf{\Omega} = 0$, finding

$$\tilde{\mu}/\mu = 1/(1-4\phi), \tag{6.1}$$

$$V/V_0 = (1-\phi)\,(1-4\phi). \tag{6.2}$$

† An alternate derivation notes that in Darcy's equation for uniform vertical flow, $0 = -dp/dz-\mu V/k-\rho g$, the pressure gradient must balance the weight of the composite material, i.e. $dp/dz = -\bar{\rho} g$. Eliminating dp/dz gives (5.41).

When $\phi = 0{\cdot}1$, the settling velocity is 20 % smaller than that given by (5.40) in the free case. The modified Brinkman results gives an even smaller settling velocity than this because of the additional constraint on the particle velocity.

It is felt that this paper has satisfactorily tied together, in one theory, Darcy's law for the flow through porous media with a law for the effective viscosity of suspensions (to call this Einstein's law is too restrictive). In both cases, the results are limited to moderate values of the volume fraction of solids. However, the theory gives a tenfold variation in both permeability and viscosity, hardly a first-order effect. To extend the results to higher concentrations requires modification of the basic approximations made, since the resulting analysis was exact. Thus, it would be necessary to account for the mutual impenetrability of the spheres, and also, probably, the friction between spheres for suspensions.

The methods used here should be applicable to many other situations. It should be possible to treat suspensions and fixed beds of spheres of different sizes (see note added in proof below). It should be possible to treat suspensions of spheres with applied torques, giving a constitutive equation with a non-symmetric stress tensor. It should be possible to account for small compressibility of the particles and fluid. It might be possible to apply the method to suspensions of non-spherical particles, but probably only for small ϕ.

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Appendix. Tentative treatment of non-uniform porous materials

We have, by (2.13),

$$0 = -\nabla(1-\phi)\,\overline{p} + \mu\nabla^2\langle \mathbf{u}\rangle - \langle \mathbf{T}\,.\,\nabla H\rangle, \tag{A 1}$$

neglecting gravitational terms for the present argument. It was observed by Saffman (1971) that, for non-uniform porosity and no flow ($T_{ij} = -\overline{p}\delta_{ij}$, $\overline{p}$ constant) the last term is

$$\langle \mathbf{T}\,.\,\nabla H\rangle = -\overline{p}\nabla\langle H\rangle = -\overline{p}\nabla(1-\phi). \tag{A 2}$$

This suggests that, in addition to the terms $A\langle \mathbf{u}\rangle + B\nabla^2\langle \mathbf{u}\rangle$ which equal $\langle \mathbf{T}\,.\,\nabla H\rangle$ in the uniform case, there should be other terms proportional to $\nabla(1-\phi)$. It would be consistent for the additional terms to combine in such a way that

$$0 = \nabla\,.\,(-\overline{p}\mathbf{I} + 2\tilde{\mu}\mathbf{D}) - \frac{\mu}{k}\langle \mathbf{u}\rangle, \tag{A 3}$$

with $\tilde{\mu}(\phi) = (\mu - B)/(1-\phi)$ and $A/(1-\phi) = \mu/k$ the same functions of non-uniform ϕ as in the uniform case. If such a form were valid for non-uniform ϕ we could integrate it over a small pill-box shaped region which covers both sides of a discontinuity in ϕ. Using the divergence theorem, and shrinking the volume to zero (so that the volume term disappears), we get the 'traction' $(-\overline{p}\mathbf{I} + 2\tilde{\mu}\mathbf{D})\,.\,\hat{\mathbf{n}}$ continuous across this interface. This can be applied to the stress calculation at a solid inclusion in a region of otherwise uniform porosity. Assume that there is a very thin boundary layer of pure fluid (porosity unity) between the solid and the porous material. The traction on the solid is then $(-p\mathbf{I}+2\mu\mathbf{D}).\hat{\mathbf{n}}$, and this may be computed from $(-\tilde{p}\mathbf{I}+2\tilde{\mu}\mathbf{D}).\hat{\mathbf{n}}$ outside the boundary layer, the two quantities being equal by continuity.

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Note added in proof. The effect of particle size distribution for suspensions of spheres turns out to be quite easy. Let $P(a)\,da$ be the probability that a particle has radius between a and $a+da$, then $nP(a)\,da$ is the expected number of particles per unit volume with radii in this range. The functional $\mathscr{F}(\langle\mathbf{u}\rangle)$ given by (5.6) is modified to

$$\mathscr{F}(\langle\mathbf{u}\rangle) = \int nP(a)\,da\left\{a^2\int_{\mathbf{r}_1=\mathbf{r}_0-a\hat{\mathbf{n}},\,\mathbf{r}_0\text{ fixed}}\langle\mathbf{T}\rangle_1.\hat{\mathbf{n}}\,d\hat{\mathbf{n}}\right\},$$

replacing n by $nP(a)\,da$ and integrating over all a. That is, we calculate the mean traction for particles of radius a and then average over particle sizes. The calculation for particles of one size proceeds through (5.33), which is then modified by the above replacement. It is noted that since a occurs in this equation only as a^3 and the volume fraction of solids is defined by

$$\phi = \tfrac{4}{3}\pi n \int a^3 P(a)\, da,$$

we get (5.34) and (5.35) as before. The only effect of the size distribution is the modification of ϕ. The particle velocity $\mathbf{V}$ however, depends explicitly on particle size and is still given by (5.38) or (5.39), particles of different sizes settling at different rates.